The Real Business Cycle Model Part 1

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Macroeconomics II

Goal

- We are going to study the so called Real Business Cycle Model.
- The model has been developed by Kydland and Prescott (1982).
- For their work (among others), they have received the Nobel price.

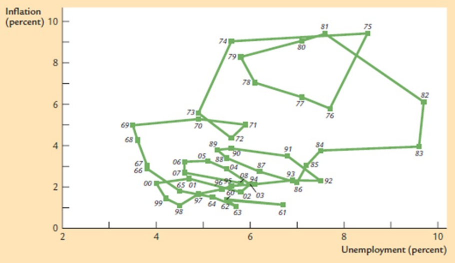
The idea

- For a long time, economists have thought about business cycles as inefficiencies.
- Hayeck (1932): Booms fed by artificially too low interest rates lead to a over-heating. A recession needs to "clean" the economy.
- Keynes (1937): Recessions result from a short fall in aggregate demand:
  - Shocks to spending.
  - Shocks to the money market.
- The dominant framework of the 70's was Phillips (1958): A negative relation between economic activity and inflation. A theory grounded in Keynesian economics with sticky prices can explain this.
- The Phillips curve provides a strong justification to use fiscal and monetary policy to smooth the business cycle.

The idea II

- Reduced-form relationships like the Phillips curve became key ingredients of policy analysis.
- This type of Macroeconomic analysis had its height in the 1970s when the FED used extensively the so called MPS model to analyze the effects of monetary policy.
- The MPS model consists of 334 equations with 188 exogenous variables!
- To make this model manageable, it assumes adaptive expectations (more on that below).

The idea III

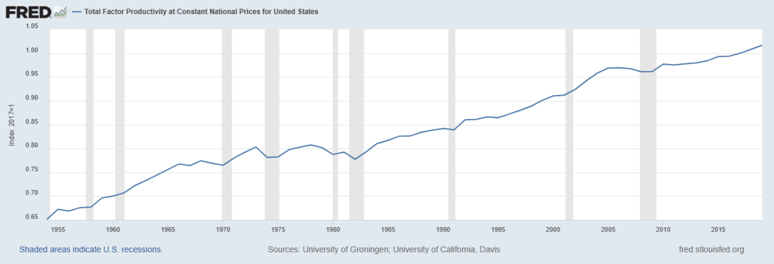


- During the 70s, economists started to realize that the reduced-form relationships such as the Philips-curve are not time-invariant.
- This has lead to a shift away from estimating reduced-form aggregate relationships and towards models of optimal behavior where agents respond to policy changes.

The idea IV

- RBC has changed our understanding of the business cycle fundamentally in two ways.
- First, it is a general equilibrium model, where agents optimize.
- Second, there are no spending shocks, sticky prices, or other market imperfections.
- Instead, households respond optimally to shocks in productivity.
- These shocks (and, hence, the cycle) are a by-product of technological advancement.
  - There is no reason for these advancements to be deterministic.
  - Hence, the economy fluctuates around a long-run trend.
- As behavior is optimal, there is no role for the government to do anything.

Recessions associated with slow TFP growth



Think about the Solow model

Suppose you have a one-time increase in TFP:

- The steady state level of capital increases.
- As output increases, $sY_t > \delta K_t \Rightarrow \Delta K_t > 0$ and this continues until the steady state is reached.
- Similarly, $C_t = (1 - s) Y_t$ increases.
- As productivity increases, wages and the interest rate are higher than in steady state.
- In the new steady state, investment and prices are again constant.

The Simplest Version

Plan

- We are going to start with the simplest version of the model.
- Households own the capital stock and possess the production technology (no need for firms).
- There is no labor supply decision.
- As all decisions are made by one entity, this is the social planner solution to the problem.

Environment

- There is a representative household who is infinitely lived and discounts the flow utility (CRRA preferences):

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}. \tag{1}$$

- It supplies inelastically one unit of labor, $H_t = 1$.
- It owns the capital stock, K_t, that depreciates at rate δ.
- It possesses a production technology for an output good: $Y_t = A_t K_t^\alpha H_t^{1-\alpha} = A_t K_t^\alpha$.

Technology

- At the heart of the RBC model lies a stochastic process for technology.
- We require a stationary environment. For simplicity, we assume technology is stationary.
- Under some assumptions, this is equivalent to a model with a deterministic trend growth rate.
- The cyclical component of technology follows:

$$\ln A_{t+1} = (1-\rho)\mu + \rho \ln A_t + \epsilon_{t+1}, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2). \tag{2}$$

- ρ guides the speed of mean reversion.
- μ simply shifts the level of technology and, thus, of output. As we do not care about the unit of measurement, we normalize $\mu = 0$ to reduce notation.

Expectations

Key to the model is that the future is uncertain:

- Households cannot make deterministic plans but only plans conditional on possible future outcomes.
- In every period t, they form expectations about the future.
- We denote these expectations by $\mathbb{E}_t$.
- But how should these expectations be formed?
- During the 60's, the typical assumption has been that people use adaptive expectations: $\mathbb{E}_t A_t = A_{t-1}$.

The rational expectation revolution

- During the 70's, economists have started to deviate from adaptive expectations.
- Adaptive expectations are inefficient and imply that households repeatedly make the same mistake.
- Instead, economists have moved to rational expectations.
- The main driving force behind this revolution has been Lucas Jr (1972).
- Which is another Nobel price winning idea.

The rational expectation revolution II

- Rational expectations assume that agents make use of all available information in an optimal way.
- They take today's state, A_t, as given and know the model including the law of motion of technology.
- Not only do they form expectations about tomorrow but about all possible future periods.
- This is complex! I need to know the probability distribution over all possible states at each point (infinite) in the future.
- Fortunately, dynamic programing simplifies this problem greatly!

The household problem

In the initial period ($t = 0$), households make a conditional plan (on possible productivity realizations) of consumption and capital choices from today to infinity:

$$\max_{C_t, K_{t+1}} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right\} \tag{3}$$

s.t.

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t \tag{4}$$

$$Y_t = A_t K_t^{\alpha} \tag{5}$$

$$I_t = K_{t+1} - (1-\delta)K_t \tag{6}$$

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1} \tag{7}$$

The maximization problem

Let λ_t be the Lagrange multiplier on the budget constraint. Hence, the Lagrangian is:

$$\Lambda_t = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \lambda_t [C_t + K_{t+1} - A_t K_t^{\alpha} - (1-\delta) K_t] \right] \right\}, \quad (8)$$

and optimal behavior is given by the first order conditions:

$$\frac{\partial \Lambda_t}{\partial C_t} = 0 \quad (9)$$

$$\frac{\partial \Lambda_t}{\partial K_{t+1}} = 0. \quad (10)$$

Optimal behavior

$$C_t^{-\gamma} = \lambda_t \tag{11}$$

$$\tag{13}$$

- (11): Marginal benefit of consumption = its marginal cost.

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$$\beta^t \lambda_t = \mathbb{E}_t \Big\{ \beta^{t+1} \lambda_{t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right) \Big\} \tag{12}$$

$$\tag{13}$$

- (11): Marginal benefit of consumption = its marginal cost.
- (12): Marginal cost of saving = its marginal benefit.
- Marginal benefit = Constrained tomorrow gets relaxed by $MPK_{t+1} + (1-\delta)$.

Optimal behavior

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$$C_t^{-\gamma} = \mathbb{E}_t \Big\{ \beta C_{t+1}^{-\gamma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right) \Big\} \tag{13}$$

- (11): Marginal benefit of consumption = its marginal cost.
- (12): Marginal cost of saving = its marginal benefit.
- Marginal benefit = Constrained tomorrow gets relaxed by $MPK_{t+1} + (1-\delta)$.
- (13) is called the Euler equation.

Optimality and expectations

$$C_t^{-\gamma} = \mathbb{E}_t \left\{ \beta C_{t+1}^{-\gamma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right) \right\} \tag{14}$$

- Note, K_{t+1} is chosen today and, hence, known today.
- However, A_{t+1} is unknown today.
- Moreover, for different realizations of A_{t+1}, the household chooses different C_{t+1} which is, thus, unknown today.
- Hence, the right hand side has the expectation operator from today. Rational expectations imply that we compute the probability distribution for each possible A_{t+1}.
- Note, the optimality condition links only period t to $t+1$. We do not require expectations over A_{t+n} $\forall n > 1$ to solve this problem.

Euler equation

Let us interpret the Euler equation:

$$C_t^{-\gamma} = \mathbb{E}_t \left\{ \beta C_{t+1}^{-\gamma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right) \right\} \tag{15}$$

At the optimum, the gain of consuming one more unit today (the marginal utility of consumption) = the gain from one more expected unit of consumption tomorrow (the expectation of marginal utility of consumption tomorrow times the expected return on savings).

Optimal behavior II

$$1 = \mathbb{E}_t \left\{ \frac{\beta C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right) \right\} \tag{16}$$

- When $\mathbb{E}_t \left\{ \frac{\beta C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right\} < 1$ the household expects consumption growth.
- In that case, $\mathbb{E}_t \left\{ \alpha A_{t+1} K_{t+1}^{\alpha-1} \right\} > \delta$.
- A high expected marginal product of capital makes me reduce consumption today relative to the future.
- Hence, a positive technology shock increases investment today.

Equilibrium

An equilibrium is a set of allocations (C_t and K_{t+1}) taking K_t, A_t, and the stochastic process for A_t as given such that the budget constrained, (4), and the optimality condition (13) hold.

Solution to the model

The solution to the model is given by the following set of equations

$$1 = \mathbb{E}_t \Big\{ \frac{\beta C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right) \Big\} \tag{17}$$

$$C_t + K_{t+1} = Y_t + (1-\delta) K_t \tag{18}$$

$$Y_t = A_t K_t^{\alpha} \tag{19}$$

$$I_t = K_{t+1} - (1-\delta) K_t \tag{20}$$

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1} \tag{21}$$

Difficulty: the Euler equation is non-linear (more on this later).

Deterministic steady state

We begin with studying the deterministic economy: $\epsilon_t = 0$ and, hence, $A_t = 1$. Let us postulate that a steady state exists with $C_t = C_{t+1} = C^{ss}$, and $K_t = K_{t+1} = K^{ss}$.

From the Euler equation:

$$K^{ss} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}. \tag{22}$$

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From the Euler equation:

$$K^{ss} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}. \tag{22}$$

Hence, we have found a steady state. Once $K_t = K^{ss}$, the Euler equation dictates that $C_t = C_{t+1}$. Note, $K^{ss} < K^{Gold}$ from the Solow model because of time discounting.

Deterministic steady state II

From the production function:

$$Y^{ss} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{\alpha}{1-\alpha}}. \tag{23}$$

From the budget constrained:

$$C^{ss} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}. \tag{24}$$

From the definition of investment:

$$I^{ss} = \delta \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}. \tag{25}$$

Linearization

- To simplify our solution of non-linear equations, we are going to use a linear approximation.
- In specific, we will use first-order Taylor approximations around the deterministic steady state: $f(x) \approx f(x^{ss}) + f'(x^{ss})(x - x^{ss})$.
- That is, we use a *purtubation* around the steady-state.
- As you know, the approximate is only good close to the point around which we approximate.
- We could use higher-order expansions to improve our approximation.

Log-linearization

In general, we could take the system as it is given. However, writing the system in logs proves to be particularly useful. The resulting solution has the interpretation of a percentage point deviation from steady state. Log-linearization follows two steps:

1. Write all variables in terms of log deviations from their deterministic steady state: $x_t = f(\ln x_t - \ln x^{ss}) = f(\hat{x}_t)$.
2. Use a first-order Taylor approximation around the deterministic steady state: $f(\hat{x}_t) \approx f(\hat{x}^{ss}) + f'(\hat{x}^{ss})(\hat{x}_t - \hat{x}^{ss})$.

We start with deriving four rules for log-lineraization that we will apply afterwards.

Log-linearization II

Using these definitions, we can write a variable x_t as:

$$x_t = x^{ss}\frac{x_t}{x^{ss}} = x^{ss}\exp(\ln x_t - \ln x^{ss}) = x^{ss}\exp(\hat{x}_t). \qquad (26)$$

Taking the Taylor expansion gives us **LI Rule 1**:

$$x_t = x^{ss}\exp(\hat{x}_t) \approx x^{ss}\exp(\hat{x}^{ss}) + x^{ss}\exp(\hat{x}^{ss})(\hat{x}_t - \hat{x}^{ss}) = x^{ss}(1+\hat{x}_t) \quad (27)$$

because $\frac{\partial \exp(\hat{x})}{\partial \hat{x}} = \exp(\hat{x})$ and $\hat{x}^{ss} = 0$.

Log-linearization III

Using the same logic, we arrive at **LI Rule 2**:

$$x_t y_t \approx x^{ss}(1+\hat{x}_t)y^{ss}(1+\hat{y}_t) \approx x^{ss}y^{ss}(1+\hat{x}_t+\hat{y}_t) \tag{28}$$

because multiplying two small numbers is approximately zero: $\hat{x}_t\hat{y}_t \approx 0$.
Moreover, we have for a constant a:

$$x_t^a = (x^{ss})^a \exp(a \ln x_t - a \ln x^{ss}) = (x^{ss})^a \exp(a\hat{x}_t). \tag{29}$$

And, hence, we arrive at **LI Rule 3**.

$$x_t^a \approx (x^{ss})^a \exp(a\hat{x}^{ss}) + (x^{ss})^a a \exp(a\hat{x}^{ss})(\hat{x}_t - \hat{x}^{ss}) = (x^{ss})^a(1+a\hat{x}_t). \tag{30}$$

Finally, **LI Rule 4** says:

$$x_t^a y_t^b \approx (x^{ss})^a(y^{ss})^b(1+a\hat{x}_t+b\hat{y}_t). \tag{31}$$

Log-linearizing investment

Investment:

$$I_t = K_{t+1} - (1-\delta)K_t \tag{32}$$

Using **LI Rule 1** yields:

$$I^{ss}(1+\hat{I}_t) = K^{ss}(1+\hat{K}_{t+1}) - (1-\delta)K^{ss}(1+\hat{K}_t) \tag{33}$$

$$\delta\hat{I}_t = \hat{K}_{t+1} - (1-\delta)\hat{K}_t. \tag{34}$$

Log-linearizing technological process

Technological progress:

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1}. \tag{35}$$

First, we slightly rewrite this equation:

$$A_{t+1} = \exp(\rho \ln A_t) \exp(\epsilon_{t+1}) \tag{36}$$

$$A_{t+1} = A_t^{\rho} \exp(\epsilon_{t+1}). \tag{37}$$

Define $y_{t+1} = \exp(\epsilon_{t+1})$, then applying **LI Rule 1** on the left side, and **LI Rule 4** the right yields:

$$(1 + \hat{A}_{t+1}) = 1 + \rho \hat{A}_t + \hat{y}_{t+1} \tag{38}$$

$$(1 + \hat{A}_{t+1}) = 1 + \rho \hat{A}_t + \ln \exp(\epsilon_{t+1}) - \ln \exp(0) \tag{39}$$

$$\hat{A}_{t+1} = \rho \hat{A}_t + \epsilon_{t+1} \tag{40}$$

because $A^{ss} = \exp(\epsilon^{ss}) = 1$.

Log-linearizing Euler equation

$$C_t^{-\gamma} = \mathbb{E}_t\Big\{\beta C_{t+1}^{-\gamma}\left(\alpha A_{t+1}K_{t+1}^{\alpha-1} + (1-\delta)\right)\Big\} \tag{41}$$

Using again **LI Rule 1** and **LI Rule 4** yields:

$$(C^{ss})^{-\gamma}(1-\gamma\hat{C}_t) = \\ \mathbb{E}_t\Big\{(C^{ss})^{-\gamma}\beta(1-\gamma\hat{C}_{t+1})\Big[1-\delta+\alpha(K^{ss})^{\alpha-1}(1+\hat{A}_{t+1}+(\alpha-1)\hat{K}_{t+1})\Big] \tag{42}$$

$$(1-\gamma\hat{C}_t) = \\ \mathbb{E}_t\Big\{(1-\gamma\hat{C}_{t+1})\Big[\beta-\beta\delta+\beta\alpha(K^{ss})^{\alpha-1}(1+\hat{A}_{t+1}+(\alpha-1)\hat{K}_{t+1})\Big]\Big\} \tag{43}$$

Insights from the Euler equation

Now substituting for the steady state capital stock:

$$(1-\gamma\hat{C}_t) = \mathbb{E}_t\Big\{(1-\gamma\hat{C}_{t+1})\Big[1+(1-\beta(1-\delta))(\hat{A}_{t+1}+(\alpha-1)\hat{K}_{t+1})\Big]\Big\} \quad (44)$$

With $\hat{C}_{t+1}\hat{A}_{t+1} \approx \hat{C}_{t+1}\hat{K}_{t+1} \approx 0$ and rearranging yields:

$$\mathbb{E}_t\hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1-\beta(1-\delta))[\mathbb{E}_t\hat{A}_{t+1} + (\alpha-1)\hat{K}_{t+1}]. \quad (45)$$

- A high capital stock tomorrow leads to low consumption growth.
- A high capital stock implies capital is relatively unproductive.
- There are little gains to further investment and, hence, consumption is high today.

Insights from the Euler equation II

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1 - \beta(1 - \delta))[\mathbb{E}_t \hat{A}_{t+1} + (\alpha - 1)\hat{K}_{t+1}]. \tag{46}$$

- High expected productivity tomorrow leads to high consumption growth.
- A high productivity makes capital more productive.
- There are high gains to further investment and, hence, consumption is low today.

Insights from the Euler equation III

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1 - \beta(1 - \delta))[\mathbb{E}_t \hat{A}_{t+1} + (\alpha - 1)\hat{K}_{t+1}]. \qquad (47)$$

- Strength depends on the elasticity of intertemporal substitution, $\frac{1}{\gamma}$.
- When households are highly willing to trade current for future consumption, productivity shocks will lead to larger responses in investment.
- Note, with a CRRA utility function, there is a one-to-one link between risk aversion and the *EIS*.

Log-linearizing budget constraint

Budget constraint:

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t \tag{48}$$

Using **LI Rule 1** gives us:

$$C^{ss}(1+\hat{C}_t) + K^{ss}(1+\hat{K}_{t+1}) = Y^{ss}(1+\hat{Y}_t) + (1-\delta)K^{ss}(1+\hat{K}_t) \tag{49}$$

$$\frac{C^{ss}}{K^{ss}}(1+\hat{C}_t) + (1+\hat{K}_{t+1}) = \frac{Y^{ss}}{K^{ss}}(1+\hat{Y}_t) + (1-\delta)(1+\hat{K}_t) \tag{50}$$

Now multiply out the constants:

$$\frac{C^{ss}}{K^{ss}}\hat{C}_t + \frac{Y^{ss}}{K^{ss}} - \delta + 1 + \hat{K}_{t+1} = \frac{Y^{ss}}{K^{ss}} + \frac{Y^{ss}}{K^{ss}}\hat{Y}_t + (1-\delta) + (1-\delta)\hat{K}_t \tag{51}$$

$$\frac{C^{ss}}{K^{ss}}\hat{C}_t + \hat{K}_{t+1} = \frac{Y^{ss}}{K^{ss}}\hat{Y}_t + (1-\delta)\hat{K}_t \tag{52}$$

Log-linearizing production function

Production function:

$$Y_t = A_t K_t^{\alpha} \tag{53}$$

Using **LI Rule 1** and **LI Rule 4** yields:

$$Y^{ss}(1 + \hat{Y}_t) = A^{ss}(K^{ss})^{\alpha}(1 + \hat{A}_t + \alpha \hat{K}_t) \tag{54}$$

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_t \tag{55}$$

$$\tag{56}$$

The equation highlights the key propagation mechanism of the RBC model. Output moves one-to-one with productivity. Additionally, it increases with the capital stock which itself is moving with productivity. The strength of this propagation depends on α.

Summarizing log-linearization

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1-\beta(1-\delta))[\mathbb{E}_t \hat{A}_{t+1} + (\alpha-1)\hat{K}_{t+1}] \tag{57}$$

$$\frac{C^{ss}}{K^{ss}}\hat{C}_t + \hat{K}_{t+1} = \frac{Y^{ss}}{K^{ss}}\hat{Y}_t + (1-\delta)\hat{K}_t \tag{58}$$

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_t \tag{59}$$

$$\delta \hat{I}_t = \hat{K}_{t+1} - (1-\delta)\hat{K}_t \tag{60}$$

$$\hat{A}_{t+1} = \rho \hat{A}_t + \epsilon_{t+1} \tag{61}$$

This is a system of five variables and five linear difference equations that we can solve (▸ Solution).

Note, with a first-order Taylor expansion, uncertainty does not affect behavior, i.e., none of the variables depends on σ_ϵ.

Parametrization

- We have seen that the model is qualitatively consistent with some basic business cycle factors.
- To understand whether it is also quantitatively consistent, we need to assign values to the different parameters.
- We will first proceed with what is called calibration: Assigning N parameter values to match N moments in the data.
- Calibrations is the simplest way but it has some drawbacks:
  - Using only some data moments wastes information.
  - There are no measures of statistical accuracy or goodness of fit.

Alternatives to calibration

Full information approach:

- Given some parameter vector p, the model generates time series for macroeconomic aggregates.
- Choose the vector p such that we maximize the likelihood that our model generates the observed data series.

GMM:

- Instead of the entire time-series, select some moments in the data.
- Given some parameter vector p, the model generates the analogous set of moments.
- Choose the vector p such that we minimize the distance between the moments observed in the data and in the model.

Calibration strategy

Kydland and Prescott (1982) suggest to use the following strategy:

- Use the parameters of the model to match long-run trends in the data. This is simply the calibration of the Neo-Classical growth model.
- The only parameters matching business cycle facts are those from the technological progress. We use these to match exactly with our model the process of TFP in the data.
- Hence, we ask how much fluctuations in macroeconomic aggregates can we explain by the amount of exogenous fluctuations in TFP from the data.

Calibration, long-run moments

- The model period is one quarter.
- A yearly real interest rate of 4%: $\beta = 0.99$.
- Match a capital share of income of 0.33: $\alpha = 0.33$.
- A capital depreciation rate of 2.5%: $\delta = 0.025$.
- Micro-estimate for risk aversion: $\gamma = 2$.

Calibration, business cycle moments

Matching moments:

- Importantly, we need to treat the model as the data, that is, apply an HP filter.
- An autocorrelation in TFP of 0.76 requires $\rho = 0.95$.
- A variance of TFP of 0.0126^2. We require $\sigma_\epsilon = 0.0095$.

Note, the autocorelation and standard deviation we use in the model are not those implied by an $AR(1)$ process: $\frac{\sigma_\epsilon}{\sqrt{(1-\rho^2)}} = 0.03$. The reason is that we HP-filter the resulting process.

Solving the model numerically

We are going to solve the model using Dynare which is an add-on program library for Matlab.

- Dynare computes for us the linearization around the steady state.
- It solves the steady state numerically.
- It simulates the economy, computes moments, and computes impulse response functions.
- It also allows for higher-order Taylor-series expansions where risk starts to matter.

The structure of Dynare

- You write your program in a so-called .mod file. Simply write it in a Matlab file and save it as a .mod file instead of a .m file.
- The program consists of 6 parts (see next slides).
- You call this file from Matlab using: dynare FILENAME noclearall

Declarations

In this part, you declare the names of your endogenous (var) and exogenous (varexo) variables, as well as, the parameters of the model.

```
%----------------------------------------------------------------------
% 1. Declarations
%----------------------------------------------------------------------

var c, k, a, y, i;
varexo e;
parameters beta, alpha, delta, rho, gamma,
       sigshock, k_init, y_init, c_init, i_init;
```

Set the parameter values

You may either set the parameter values directly in Dynare, or load them from a Matlab file as I do here:

```
%----------------------------------------------------------------------
% 2. Parameter values
%----------------------------------------------------------------------

%!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
% Below load and set all the necessary parameter values
%!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

load parametervalues;
set_param_value('beta',par.beta);
set_param_value('alpha',par.alpha);
set_param_value('delta',par.delta);
set_param_value('rho',par.rho);
set_param_value('gamma',par.gamma);
set_param_value('sigshock',par.sigshock);
set_param_value('k_init',par.k_init);
set_param_value('y_init',par.y_init);
set_param_value('c_init',par.c_init);
set_param_value('i_init',par.i_init);
```

Model equations

Now, you need to write the equilibrium equations of your model. Note, here I write all variables in exp so that Dynare linearizes around logs of the variables. That is, the level of consumption is actually exp(c):

```
%----------------------------------------------------------------------------
% 3. Model equations
%----------------------------------------------------------------------------

%!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
% Below fill in the model block
%!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

model;
exp(c)+exp(k) = exp(y)+(1-delta)*exp(k(-1));
exp(y) = exp(a)*(exp(k(-1))^alpha);
a = rho*a(-1)+e;
exp(c)^(-gamma) = beta*exp(c(+1))^(-gamma)*(alpha*exp(a(+1))*(exp(k)^(alpha-1))+1-delta);
exp(i) = exp(k) - (1-delta)*exp(k(-1));
end;
```

Model equations II

Dynare has as convention to time the variable on when it is *decided*. As K_{t+1} has been already decided in t, it is dated with t. In contrast, C_{t+1} is decided in $t+1$ and, hence, is dated with $+1$:

```
%----------------------------------------------------------------------------
% 3. Model equations
%----------------------------------------------------------------------------

%!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
% Below fill in the model block
%!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

model;
exp(c)+exp(k) = exp(y)+(1-delta)*exp(k(-1));
exp(y) = exp(a)*(exp(k(-1))^alpha);
a = rho*a(-1)+e;
exp(c)^(-gamma) = beta*exp(c(+1))^(-gamma)*(alpha*exp(a(+1))*(exp(k)^(alpha-1))+1-delta);
exp(i) = exp(k) - (1-delta)*exp(k(-1));
end;
```

Steady state

Next, you need to compute the steady state. Dynare uses a non-linear equation solver (Newton). Here, you need to provide some initial values:

```
%-----------------------------------------------------
% 4. Steady states
%-----------------------------------------------------

initval;

k = k_init;
y = y_init;
c = c_init;
i = i_init;
a = 0;

end;
steady;
```

Exogenous shocks

Next, we need to specify the exogenous shocks which is just one in our case:

```
%----------------------------------------------
% 5. Shocks
%----------------------------------------------

shocks;
var e; stderr sigshock;
end;
```

Solution

Finally, you need to tell Dynare to compute the solution to the model. I tell Dynare here to apply a HP-filter and a first-order Taylor series approximation:

```
%---------------------------------------------
% 6. Solution
%---------------------------------------------

stoch_simul(hp_filter= 1600, order = 1);
```

Understanding the Dynare output

First, Dynare provides you with the solution of steady-state variables (in my code the log steady state):

```
STEADY-STATE RESULTS:

c          0.835782
k          3.34457
a          0
y          1.10371
i          -0.344308
```

Understanding the Dynare output II

Next, Dynare gives us the policy functions. The constant is simply the steady state:

POLICY AND TRANSITION FUNCTIONS

	c	k	a	y	i
Constant	0.835782	3.344571	0	1.103709	-0.344308
k(-1)	0.440543	0.974256	0	0.330000	-0.029780
a(-1)	0.345784	0.072913	0.950000	0.950000	2.916523
e	0.363983	0.076751	1.000000	1.000000	3.070024

For example, given my log definition of variables, the policy function for consumption is written as

$$\hat{C}_t = a_1 \hat{K}_t + a_2 \hat{A}_{t-1} + a_3 \epsilon_t. \tag{62}$$

Understanding the Dynare output III

Next, we receive some summary statistics computed on the HP-filtered data:

THEORETICAL MOMENTS (HP filter, lambda = 1600)

VARIABLE	MEAN	STD. DEV.	VARIANCE
c	0.8358	0.0047	0.0000
k	3.3446	0.0034	0.0000
a	0.0000	0.0124	0.0002
y	1.1037	0.0124	0.0002
i	-0.3443	0.0380	0.0014

Understanding the Dynare output IV

Then, Dynare provides the correlation among HP-filtered variables:

MATRIX OF CORRELATIONS (HP filter, lambda = 1600)

Variables	c	k	a	y	i
c	1.0000	0.5298	0.9475	0.9725	0.9466
k	0.5298	1.0000	0.2307	0.3178	0.2281
a	0.9475	0.2307	1.0000	0.9959	1.0000
y	0.9725	0.3178	0.9959	1.0000	0.9956
i	0.9466	0.2281	1.0000	0.9956	1.0000

Understanding the Dynare output V

Finally, we have the autocorrelation structure of HP-filtered variables:

COEFFICIENTS OF AUTOCORRELATION (HP filter, lambda = 1600)

Order	1	2	3	4	5
c	0.7528	0.5341	0.3447	0.1845	0.0524
k	0.9603	0.8640	0.7306	0.5759	0.4128
a	0.7133	0.4711	0.2711	0.1098	-0.0163
y	0.7195	0.4810	0.2826	0.1216	-0.0055
i	0.7131	0.4710	0.2709	0.1096	-0.0165

Comparing model and data

	Data			
	Y	C	I	TFP
Std. %	1.61	1.25	7.27	1.25
ACR(1)	0.78	0.68	0.78	0.76
	Model			
Std. %	1.24	0.47	3.8	1.24
ACR(1)	0.72	0.75	0.71	0.71

Comparing model and data

Correlations

	Y	C	I	TFP
		Data		
Y	1			
C	0.78	1		
I	0.83	0.67	1	
TFP	0.79	0.71	0.77	1
		Model		
Y	1			
C	0.97	1		
I	1	0.95	1	
TFP	1	0.95	1	1

Comparing model with data

The successes:

- The model replicates broad co-movement of all macroeconomic aggregates.
- The autocorrelations of all aggregates are of the right size.
- Investment is much more volatile than other aggregates.
- Consumption is less volatile than output.
- The correlation is weakest between consumption and other aggregates suggesting consumption smoothing.

The misses:

- The model has too little propagation: Output just as volatile as TFP.
- The co-movements between the variables is too strong.

Monte-Carlo simulation and impulse responses

- Dynare computes so called impulse responses.
- You may want to do this yourself.
- Dynare can compute moments based on simulations of the model.
- Again, you may want to simulate the economy yourself.
- For this, we use the policy functions that Dynare has computed.

State-space representation

- Dynare saves the policy functions in their so called state-space form.
- Let S_t be a vector of the states, i.e. $\hat{K}_t$ and $\hat{A}_t$.
- Let X_t be a vector of the controls, i.e. $\hat{C}_t$, $\hat{Y}_t$ and $\hat{I}_t$.

$$S_t = AS_{t-1} + B\epsilon_t \tag{63}$$

$$X_t = CS_{t-1} + D\epsilon_t \tag{64}$$

Understanding Dynare

- Dynare stores these matrices.
- Matrices A and C are stored in $oo\_.dr.ghx$.
- Matrices B and D are stored in $oo\_.dr.ghu$.
- The order of the variables is not as we have defined variables. The vector $oo\_.dr.inv\_order\_var$ provides the mapping from our order of variables to the order that Dynare has stored the variables.

Retrieving the matrices

```
%order that variables are declared
p_c = 1;
p_k = 2;
p_a = 3;
p_y = 4;
p_i = 5;

%dynamics of states to states
A = [oo_.dr.ghx(oo_.dr.inv_order_var(p_k),:);
    oo_.dr.ghx(oo_.dr.inv_order_var(p_a),:)];

%dynamics of shocks to states
B = [oo_.dr.ghu(oo_.dr.inv_order_var(p_k),:);
    oo_.dr.ghu(oo_.dr.inv_order_var(p_a),:)];

%dynamics of states to controlls
C = [oo_.dr.ghx(oo_.dr.inv_order_var(p_c),:);
    oo_.dr.ghx(oo_.dr.inv_order_var(p_y),:);
    oo_.dr.ghx(oo_.dr.inv_order_var(p_i),:)];

%dynamics of shocks to controlls
D = [oo_.dr.ghu(oo_.dr.inv_order_var(p_c),:);
    oo_.dr.ghu(oo_.dr.inv_order_var(p_y),:);
    oo_.dr.ghu(oo_.dr.inv_order_var(p_i),:)];
```

Impulse response functions

- Using the state-space representation also allows us to compute what is called impulse responses.
- This is the dynamic behavior of all variables that have been in steady state and receive a one-time exogenous shock (1 std).
- After this one shock, no further shocks occur and the economy will eventually return to its steady state.
- In period one, this is simply

$$S_1 = B\epsilon_1 \tag{65}$$

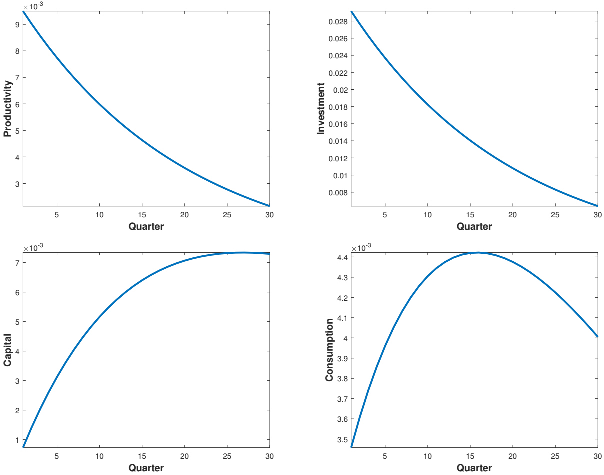
$$X_1 = D\epsilon_1 \tag{66}$$

- Afterwards, we have with no further shocks:

$$S_t = AS_{t-1} \tag{67}$$

$$X_t = CS_{t-1} \tag{68}$$

Impulse response functions



Impulse response functions II

- After an increase in productivity, investment increases.
- This leads to a slow build-up in capital.
- Higher *TFP* and capital increase output.
- As *MPK* is high initially, consumption rises by less than investment.
- Over time, as *MPK* declines, consumption increases.
- As output returns to its initial level, consumption starts to decline again at some point.
- In total, consumption is relatively smooth.

Simulating the economy

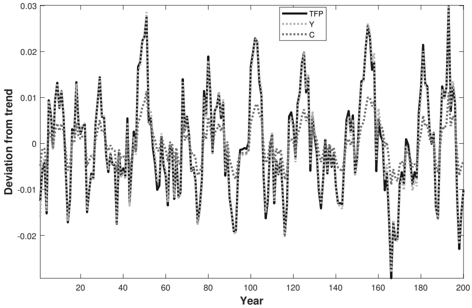
- The idea is to draw a long series of random numbers for the productivity shocks.
- Given these shocks, we can compute the resulting macroeconomic aggregates.

```
randn('seed',2557)
e = par.sigshock*randn(par.T,1);
Ssim = zeros(2,par.T); %states
Xsim = zeros(3,par.T); %controls

Ssim(:,1) = B*e(1);
for t = 2:par.T
    Ssim(:,t) = A*Ssim(:,t-1)+B*e(t);
    Xsim(:,t) = C*Ssim(:,t-1)+D*e(t);
end

% HP filter
[~,hp.k] = hpfilter(Ssim(1,:)',1600);
[~,hp.a] = hpfilter(Ssim(2,:)',1600);
[~,hp.c] = hpfilter(Xsim(1,:)',1600);
[~,hp.y] = hpfilter(Xsim(2,:)',1600);
[~,hp.i] = hpfilter(Xsim(3,:)',1600);
```

Results of the simulation



Back to the beginning

- So far, we have solved the model using (log)-linearization.
- We are now going to solve the model globally.
- In particular, we are going to use value function iteration.
- Importantly, now uncertainty is going to matter.

The recursive formulation

You have already seen the recursive formulation:

$$V(K,A) = \max_{C,K'} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \mathbb{E}_t V(K',A') \right\} \tag{69}$$

s.t.

$$\begin{aligned} C_t + K_{t+1} &= Y_t + (1-\delta)K_t \\ Y_t &= A_t K_t^{\alpha} \\ I_t &= K_{t+1} - (1-\delta)K_t \\ \ln A_{t+1} &= \rho \ln A_t + \epsilon_{t+1} \end{aligned}$$

Solving the recursive formulation

- We have to parametrize $\mathbb{E}_t$.
- We assume productivity follows a continuous $AR(1)$ process. To put it in a computer, we need to discretize it.
- The method most commonly used for this is the Tauchen algorithm.

Value function iteration algorithm

- Construct a grid for capital $K_i = \{k_1, k_2, ...k_{N_k}\}$.
- Construct a grid for productivity $A_j = \{A_1, A_2, ...A_{N_s}\}$ and corresponding transition matrix P.

1. Guess a continuous/increasing value function $V^0(K_i, A_j)$ of dimension $N_k \ X \ N_a$.
2. Solve $V^n(K, A) = \underset{C,K'}{max}\Big\{u(c) + \beta P(A, A')V^{n-1}(K', A')\Big\}$.
3. Replace last iteration guess by new solution $V^{n-1} = V^n$.
4. Iterate until $|V^n - V^{n-1}| < crit$.

Appendix

Solving the system (general form)

Blanchard and Kahn (1980) suggest one possible solution technique that first writes the problem in VAR form:

$$A_1 \begin{bmatrix} \mathbb{E}_t X_{t+1} \\ \mathbb{E}_t Y_{t+1} \end{bmatrix} = A_0 \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + aZ_{t+1}, \tag{70}$$

where X_t are the state variables ($\hat{K}_t$, $\hat{A}_t$), Y_t are the forward-looking controls (or jumpers, $\hat{C}_t$), and Z_{t+1} are mean zero shocks. Note, for simplicity, I omit output, $\hat{Y}_t$, and investment, $\hat{I}_t$, which can be derived from the other variables.

Solving the system (our case)

$$\begin{bmatrix} (1-\alpha)\frac{1}{\gamma}(1-\beta(1-\delta)) & -\frac{1}{\gamma}(1-\beta(1-\delta)) & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{K}_{t+1} \\ \mathbb{E}_t \hat{A}_{t+1} \\ \mathbb{E}_t \hat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \alpha \frac{Y^{ss}}{K^{ss}} + 1 - \delta & \frac{Y^{ss}}{K^{ss}} & -\frac{C^{ss}}{K^{ss}} \\ 0 & \rho & 0 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{A}_t \\ \hat{C}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \epsilon_{t+1} \quad (71)$$

Back to the general case

Define $A = A_1^{-1} A_0$ and $R = A_1^{-1} a$:

$$\begin{bmatrix} \mathbb{E}_t X_{t+1} \\ \mathbb{E}_t Y_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + R Z_{t+1}, \tag{72}$$

Blanchard and Kahn (1980) show that

- a unique solution exists iff the number of eigenvalues of A lying outside the unit circle (unstable roots) is equal to the number of jumpers.
- no solution exists if there are too many unstable eigenvalues.
- infinitely many solutions exist if there are too few unstable eigenvalues.

Back

Root Finding

We are looking for $x_1, ..., x_n$ such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \Leftrightarrow \begin{cases} 0 = f^1(x_1, ..., x_n) \\ ... \\ 0 = f^n(x_1, ..., x_n) \end{cases} \tag{73}$$

For simplicity, let us start with the univariate case:

$$f(x) = 0. \tag{74}$$

Newton-Raphson Method

- Newton method uses first order approximation to the function.
- First order approximation around guess x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

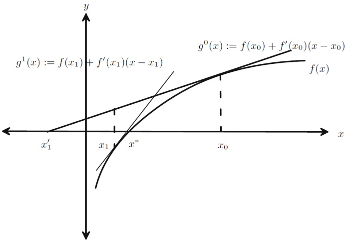
- Setting $f(x) = 0$ and solving for x gives new guess:

$$x' = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The tangent intersects the x-axis.

- This requires **numerical differentiation** (in one second)!

Modified Newton-Raphson Method



- When the objective function is close to flat around x^0, the linear approximation may lead to a poor prediction.

- Function may not be defined at x'.

  Reformulating the problem is often possible.

- The Modified Newton-Raphson Method updates slowly $\lambda \in [0, 1]$:
  $x' = x_0 - \lambda \frac{f(x_0)}{f'(x_0)}$.

Multivariate case

The method can be extended straightforward to the multivariate case:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \Leftrightarrow \begin{cases} 0 = f^1(x_1, ..., x_n) \\ ... \\ 0 = f^n(x_1, ..., x_n) \end{cases}$$

Define the Jacobian:

$$\mathbf{J}(\mathbf{a}) = \begin{bmatrix} f_1^1 & f_2^1 & f_3^1 & \dots & f_n^1 \\ f_1^2 & f_2^2 & f_3^2 & \dots & f_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^n & f_2^n & f_3^n & \dots & f_n^n \end{bmatrix}, \quad f_j^i = \frac{\partial f^i(\mathbf{x})}{\partial x_j}$$

Multivariate case II

Approximate

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0) + \mathbf{J}(\mathbf{x_0})(\mathbf{x} - \mathbf{x}_0),$$

with solution

$$\mathbf{x}' = \mathbf{x}_0 - \lambda \mathbf{J}(\mathbf{x}_0)^{-1}\mathbf{f}(\mathbf{x}_0).$$

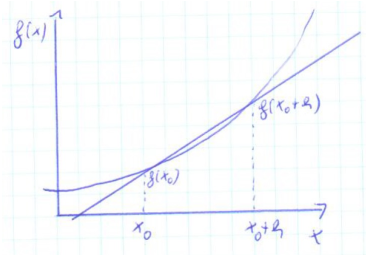
Numerical Differentiation

For this algorithm, we need to compute

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- Simplest method called one sided approximation:

  $f'(x) \approx \frac{f(x+h)-f(x)}{h}$. Slope error proportional to h

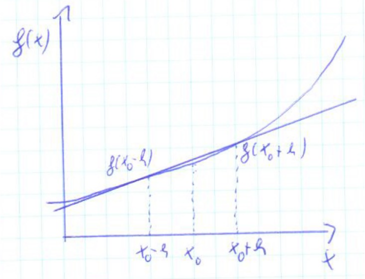


Numerical Differentiation II

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Two sided approximation:

  $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$. Slope error proportional to h^2.



Back

Markov Chains

- Idea: Use a first-order Markov chain to approximate the continuous $AR(1)$ process.

- A Markov-chain is characterized by a discrete grid s_i , $i = 1 : N$ and a transition probability matrix P giving the probability to move from point i to j, p_{ij}.

- Hence, $S_t = PS_{t-1}$ gives the probability distribution over states in recursive form.

Markov Approximation of $AR(1)$

Consider the generalized $AR(1)$ process:

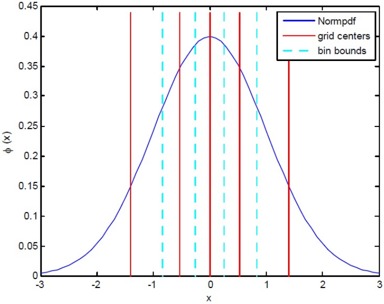
$$A_t = (1-\rho)\mu + \rho A_{t-1} + \epsilon_t \quad \epsilon_t \sim N(0, \sigma^2)$$

- The process has a mean μ.
- We impose normality for the shock distribution!
- Ergodic distribution is $N(\mu, \sigma_{AR}^2)$ with $\sigma_{AR}^2 = \frac{\sigma^2}{1-\rho^2}$.

Tauchen (1986) Algorithm

- Idea: Partition ergodic distribution in N bins and choose points in bins *representing* those bins.
- Choose N bins such that each is equally likely.

Graphical Representation



Create Bins

Choose boundaries, b_i , of bins, S_i , according to:

$$P(b \in S_i) = \Phi\Big(\frac{b_{i+1} - \mu}{\sigma_{AR}}\Big) - \Phi\Big(\frac{b_i - \mu}{\sigma_{AR}}\Big) = \frac{1}{N}.$$

Hence,

$$\Phi\Big(\frac{b_{i+1} - \mu}{\sigma_{AR}}\Big) = \frac{i}{N}.$$

or

$$b_{i+1} = \sigma_{AR}\Phi^{-1}\Big(\frac{i}{N}\Big) + \mu.$$

Centers of Bins

Next is to choose a representative element, s_i , for each bin:

$$s_i = (s | s \in S_i).$$

One can show that with a normal distribution this is:

$$s_i = N\sigma_{AR}\left[\phi\left(\frac{b_i - \mu}{\sigma_{AR}}\right) - \phi\left(\frac{b_{i+1} - \mu}{\sigma_{AR}}\right)\right] + \mu.$$

Transition Probabilities

We need to know the transition matrix. E.g., what is the probability for $s \in S_i$ to move to $s' \in S_j$?

We need

$$b_j \leq \rho s + (1-\rho)\mu + \epsilon$$
$$b_{j+1} \geq \rho s + (1-\rho)\mu + \epsilon$$

Thus

$$\epsilon \in [b_j - \rho s - (1-\rho)\mu, b_{j+1} - \rho s - (1-\rho)\mu].$$

Simplified Tauchen Algorithm

$$p_{i,j} = P(s' \in S_j | s \in S_i) =$$
$$\Phi\left(\frac{b_{j+1} - \rho s_i - (1-\rho)\mu}{\sigma}\right) - \Phi\left(\frac{b_j - \rho s_i - (1-\rho)\mu}{\sigma}\right).$$

- There is a more accurate formulation where all points in S_i are taken into account, not only s_i.
- This requires integrating over the relevant part of the distribution and weighting by the probability of each occurrence.

Back

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